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SEMI-INVARIANT FORM OF EQUILIBRIUM STABILITY CRITERIA IN CRITICAL CASES*

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Expressions are given for the first coefficients of the normal form of the equations in the neutral equilibrium manifold of an autonomous system in the basic critical cases, in terms of the neutral eigenvectors of the linearized system and its conjugate. Using the reduction principle /1/, the stability criteria previously obtained by various authors in critical cases take an explicit form, which is very convenient for calculations and does not involve the restriction of finite dimensionality.

If the equilibrium stability spectrum of a non-linear autonomous system of differential equations contains a point on the imaginary axis (neutral spectrum), while its remaining part is inside the left half-plane, the answers to the questions on stability, asymptotic stability, and instability, depend on the nature of the non-linearity and cannot be given by an analysis of just the linearized system. There have been studies of these so-called critical cases, starting with Lyapunov's calssical work /2, 3/. The results so far obtained have been classified and considerably supplemented in /4/**,(**See also, E.E., Shnol' and L.G. Khazin, "On the stability of stationary solutions of general systems of differential equations close to critical cases", Preprint In-ta prikl. matem. AN SSSR, Moscow, 1979, No.91; L.G. Khazin and E.E. Shnol', "On soft and hard loss of stability of stationary solutions of differential equations", Preprint In-ta prikl. matem. AN SSSR, Moscow, 1979, No.128.) where the critical cases are classified according to the co-dimensionality of the degeneracy in the space of all possible holomorphic systems. In the case of weak degeneracy, i.e., in all cases of co-dimensionality of degeneracy less than or equal to two, with a single exception (two pairs of pure imaginary eigenvalues, resonance 1: 3) of co-dimensionality three, there are explicit critieria for asymptotic stability and instability. They are expressed by strict inequalities for several Taylor coefficients of the right-hand side of the so-called model system. The latter splits off from the normal form of the initial system when the leading terms are discarded, and its order is equal to the total multiplicity of the neutral spectrum (dimensionalty of neutral subspace). In the long run, the application of criteria to a specific system amounts basically to evaluating the coefficients of the model system.

Below, we give explicit expressions for these coefficients in "semi-invariant" form, i.e., we use the Jordan basis only in the neutral subspace, and not in the entire space. Nowhere in these expressions does the system dimensionality appears, so that they can also be used for infinite-dimensional problems, e.g., for systems of partial differential equations.

The impossibility of reducing a system to normal form in the general infinite-dimensional case tells us that it likewise cannot be done for finite-dimensional problems. We can expect that, even for systems of ordinary differential equations, the simplest approach is to use these expressions possibly for computer calculations. The main part of our paper is an evaluation of the neutral root vectors of the linearized system and its conjugate, and also the solution of certain inhomogeneous algebraic equations. Of course, we can only study the multiplicity of the spectrum and the corresponding Jordan structure of the system in the neutral subspace, in those cases when the system contains parameters, for individual values of which (which have to be determined) degeneracy occurs. In other cases, incidentally, the multiplicity is defined by the general properties of the system.

The two essential features of our present study are application of the reduction principle of /1/, according to which it suffices to consider the contraction of the system to the neutral manifold of equilibrium, and the reduction of the complete system to the partially normal form, in which we discard only those non-resonance terms which contain netural variables only, in all the equations at once.

The transition to a system in the neutral manifold is justified for finite-dimensional systems by the reduction principle of /1/, and may also be justified for those infinite-dimensional systems for which a netral manifold theorem holds and for which there is a suitable reduction principle.

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1. Formulation of the problem and description of method. Consider an autonomous real differential equation in R^n with zero equilibrium

$$u^{\cdot} = f(u) = Au + g(u),\ g(u) = K_2 u^2 + K_3 u^3 + \ldots,\ f(0) = 0 \tag{1.1}$$

Here, $u \to f(u)$ is an analytic vector field, defined in the neighbourhood of $0 \in R^n$ (it will be clear from what follows that each of the following results holds for $f \in C^k$, $k \leqslant 3$), $A: R^n \to R^n$ is a linear operator; $K_m u^m$ is a homogeneous operator $u \to K_m u^m$ of degree m, acting in R^n, and defined by the m-linear mapping $(u_1, u_2, \ldots, u_m) \to K_m (u_1, u_2, \ldots, u_m)$, where $K_m u^m = K_m (u, u, \ldots, u)$; $u_1, u_2, \ldots, u_m, u \in R^n$. A symmetric $K_m (u_1, u_2, \ldots, u_m)$ can be taken, though this is not essential.

We assume that the spectrum $\sigma(A)$ of operator A can be written as the union of pairs of spectral sets $\sigma_1(A)$ and $\sigma_0(A)$, where $\sigma_1 = \sigma_1(A) = \{\lambda : \lambda \in \sigma(A),\ \mathrm{Re}\lambda < 0\}$ lies strictly in the left half-plane, and $\sigma_0 = \sigma_0(A) = \{\lambda : \lambda \in \sigma(A),\ \mathrm{Re}\lambda = 0\}$ lies on the imaginary axis.

Corresponding to spectral sets σ_0 and σ_1 we have the spectral subspaces X_0, X_1 and spectral projectors P_0, P_1,

$$P_i = \int_{\Gamma_i} (\lambda I - A)^{-1} d\lambda, \quad i = 0, 1$$

Here, the contour $\Gamma_i = \partial D_i$ $(D_0 \cap D_1 = \varnothing)$ is the smooth boundary of the bounded domain D_i of the complex plane, while $\sigma_i \subset D_i$.

We shall call X_0 the neutral subspace of the operator A, and X_1 the stable subspace.

It follows from /1/ that, in a neighbourhood $V \subset R^n$ of the point 0 there is an invariant submanifold $M_0 \subset V$ of Eq.(1.1), which touches the netural subspace X_0 at the point 0. Here, M_0 is the graph of a mapping $F: V_0 \to X_1$ $(V_0 \subset V \cap X_0)$ of the neighbourhood of zero of the subspace X_0 into X_1.

The invariant submanifold M_0 will be called the netural manifold (it is sometimes called the central manifold /5/).

The problems of stability of the zero equilibria of Eq.(1.1) and of contraction of this system *onto* M_0 *are equivalent (the reduction principle /1/)*.

We will write (1.1) as the system

$$\begin{aligned} x_i^{\cdot} &= P_i f_*(x_0, x_1) = A_i x_i + P_i g_*(x_0, x_1),\ i = 0,1 \\ x_i &= P_i u \in X_i,\ f_*(x_0, x_1) = f(x_0 + x_1) \\ g_*(x_0, x_1) &= g(x_0 + x_1) \end{aligned} \tag{1.2}$$

where A_i is the contraction of the operator A onto X_i. We introduce new variables $\xi_0 = x_0$, $\xi_1 = x_1 - F(x_0)$. The neutral manifold is given by the equation $\xi_1 = 0$, and the system (1.2) takes the form

$$\begin{aligned} \xi_0^{\cdot} &= P_0 f_*(\xi_0, \xi_1 + F(\xi_0)) \\ \xi_1^{\cdot} &= (P_1 - F'(\xi_0) P_0) f_*(\xi_0, \xi_1 + F(\xi_0)) \end{aligned} \tag{1.3}$$

By the reduction principle, it suffices to study the stability of the zero equilibrium of the equation in the neutral manifold M_0 $(\xi_1 = 0)$

$$\xi_0^{\cdot} = P_0 f_*(\xi_0, F(\xi_0)) \tag{1.4}$$

The *mapping* F *can be sought as the Taylor series*

$$F(\xi_0) = F_2 \xi_0^2 + F_3 \xi_0^3 + \ldots \tag{1.5}$$

Here, the mapping $F_i: \xi_0 \to F_i \xi_0^i$ $(i = 2, 3, \ldots)$ is a symmetric vector-valued form of the i-th degree in X_0 with values in X_1.

Series (1.5) is regarded as asymptotic. We know that, though it may prove to be divergent /1/, in each of the critical cases considered below we only need to known its first few terms.

To find F we obtain from the invariance condition the equation

$$P_1 f_*(\xi_0, F(\xi_0)) = F'(\xi_0) P_0 f_*(\xi_0, F(\xi_0)) \tag{1.6}$$

Substituting series (1.5) into (1.6) and equating terms in like powers on each side, we obtain a system of equations for the forms F_i $(i = 2, 3, \ldots)$.

Replacing $y' = \xi_0 - G(\xi_0)$ $(G: V_0 \to X_0)$ we reduce Eq.(1.4) to the normal form /6/ up to a certain order.

In each of the critical cases considered below, the normal form is defined uniquely. If it is not uniquely defined, as e.g., in the case of a Jordan cell, it is a matter of indifference which definition is chosen, provided that there is a stability criterion for it.

The reduction of Eq.(1.4) to the normal form is sometimes greatly simplified if we seek F *and* G *in the form* $F = P_1 T$, $G = P_0 T$, *and define the mapping* $T: V_0 \to V$ in such a way that the change of variables $u' = u - T(x_0)$ in Eq.(1.1) or in the equivalent system (1.3) decreases the non-resonant terms which contain x_0.

In each critical case considered below, the results are given according to the following scheme:

1) the conditions for degeneracy of the linear part are found. The neutral spectrum σ_0 is found and the projector P_0 is introduced;

2) the equation in the neutral manifold M_0 is given in the normal form. Only those terms of the expansion which participate in the stability criterion are written;

3) the stability criterion in the sense of /4/ is given; i.e., the set of strict inequalities which ensure the asymptotic stability or instability of the zero resonace *. (*For more details and some cases not considered here, see: L.G. Kurakin and V.I. Yudovich, "On critical cases of Lyapunov stability," Rostov on Don, 1985, Dep. at VINITI 17.07.85; No.5132-85)

We use the notation below: μ is the co-dimensionality of degeneracy of the critical case, $K_2^0(u, v) = K_2(u, v) + K_2(v, u)$, $K_3^0(u^2, v) = K_3(u, u, v) + K_3(u, v, u) + K_3(v, u, u)$.

2. The case of a simple zero eigenvalue. Neutral spectrum $\sigma_0 = \{0\}$. Here, 0 is a simple eigenvalue, and the corresponding eigenvectors of the operator A and its adjoint A^* are φ, Φ

$$A\varphi = 0,\ A^*\Phi = 0,\ (\varphi, \Phi) = 1$$

The projector onto the neutral subspace X_0 is $P_0u = (u, \Phi)\varphi$.

The equation on the neutral manifold in the normal form is

$$\begin{aligned}
&x_0^{\cdot} = a_2x_0^2 + a_3x_0^3 + a_4x_0^4 + \ldots \\
&a_2 = (K_2\varphi^2, \Phi),\ a_3 = (K_2^0(\varphi, z_2) + K_3\varphi^3, \Phi) \\
&a_4 = (K_2^0(\varphi, z_1) + K_2z_2^2 + K_3^0(\varphi^2, z_2) + K_4\varphi^4, \Phi) \\
&z_1 = -A_1^{-1}(L_2^0(\varphi, z_2) + L_3\varphi^3 - 2a_2z_2),\ z_2 = -A_1^{-1}L_2\varphi^2
\end{aligned}$$

where A_1, L_i, L_2^0 are the contractions onto X_1 of respectively operators $A, (I - P_0)K_i, (I - P_0)K_2^0$.

Stability criterion. Let a_k be the first non-zero Lyapunov quantity (case $\mu = k$). We know (Lyapunov) that, for even k, the zero equilibrium is unstable, and for odd k, it is unstable when $a_k > 0$ and stable when $a_k < 0$.

3. The case of one zero and a pair of purely imaginary eigenvalues. Neutral spectrum $\sigma_0 = \{0, \pm i\omega\}$, $\omega > 0$. These eigenvalues are simple, while the corresponding eigenvectors of the operators A, A^* are $\{\varphi, \varphi_1, \varphi_1^*\}$, $\{\Phi, \Phi_1, \Phi_1^*\}$:

$$\begin{aligned}
&A\varphi = 0,\ A\varphi_1 = i\omega\varphi_1,\ A\varphi_1^* = -i\omega\varphi_1^* \\
&A^*\Phi = 0,\ A^*\Phi_1 = -i\omega\Phi_1,\ A^*\Phi_1^* = i\omega\Phi_1^* \\
&(\varphi, \Phi) = (\varphi_1, \Phi_1) = (\varphi_1^*, \Phi_1^*) = 1
\end{aligned}$$

The projector onto the neutral subspace X_0 is

$$P_0u = (u, \Phi)\varphi + (u, \Phi_1)\varphi_1 + (u, \Phi_1^*)\varphi_1^*$$

The equation in the neutral manifold in the normal form is

$$\begin{aligned}
&y^{\cdot} = a_{11}y^2 + a_{12}y_1y_1^* + a_{13}y^3 + \ldots,\ y_1^{\cdot} = i\omega y_1 + a_{21}yy_1 + \ldots \\
&y = (y', \Phi),\ y_1 = (y', \Phi_1),\ a_{11} = (K_2\varphi^2, \Phi) \\
&a_{12} = (K_2^0(\varphi_1, \varphi_1^*), \Phi),\ a_{21} = (K_2^0(\varphi, \varphi_1), \Phi_1) \\
&a_{13} = (K_2^0(z, \varphi) + K_3\varphi^3, \Phi),\ z = -A^{-1}(K_2\varphi^2 - a_{11}\varphi)
\end{aligned}$$

Stability criterion. If $a_{11} \neq 0$ ($\mu = 2$), the equilibrium is unstable. If $a_{11} = 0$ ($\mu = 3$), we have stability under the conditions $a_{12}\mathrm{Re}\,a_{21} < 0$, $a_{13} < 0$, and instability when at least one condition is strictly violated.

4. The case of n-tuple zero eigenvalue (Jordan cell). Neutral spectrum $\sigma_0 = \{0\}$. The eigenvalue of 0-multiplicity n, the index $\nu(0) = n$, the corresponding eigenvectors of the operators A, A^* are φ_1, Φ_1. Let $\{\varphi_2, \varphi_3, \ldots, \varphi_n\}$ be associated vectors of the operator A, and $\{\Phi_2, \Phi_3, \ldots, \Phi_n\}$ associated vectors of A^*, corresponding to the zero eigenvalue

$$\begin{aligned}
&A\varphi_1 = 0,\ A\varphi_2 = \varphi_1,\ A\varphi_3 = \varphi_2, \ldots, A\varphi_n = \varphi_{n-1} \\
&A^*\Phi_1 = 0,\ A^*\Phi_2 = \Phi_1,\ A^*\Phi_3 = \Phi_2, \ldots,\ A^*\Phi_n = \Phi_{n-1} \\
&(\varphi_j, \Phi_{n-j+1}) = 1,\ (\varphi_k, \Phi_l) = 0,\ j = 1, 2, \ldots, n;\ k + l \neq n + 1
\end{aligned}$$

The projector onto the neutral subspace X_0 is

$$P_0u = \Sigma\,(u, \Phi_{n-k+1})\varphi_k;\ k = 1, 2, \ldots, n$$

The equation in the neutral manifold in the normal form is

$$y_j^{\cdot} = y_{j+1} + \ldots, \; y_n^{\cdot} = (K_2\varphi_1^2, \; \Phi_1)\, y_1^2 + \ldots$$
$$j = 1, \; 2, \ldots, \; n-1; \; y_k = (u, \; \Phi_{n-k+1}), \; k = 1, 2, \ldots, \; n$$

Stability criterion. If $(K_2\varphi_1^2, \; \Phi_1) \neq 0$ $(\mu = n)$, the zero equilibrium in unstable.

5. The case of n pairs of purely imaginary eigenvalues without resonances. The neutral spectrum $\sigma_0 = \{\pm i\omega_1, \ldots, \pm i\omega_n\}$, $\omega_j > 0$, $j = 1, 2, \ldots, n$, where there are no resonances of second and third order. These eigenvalues are simple, while the corresponding eigenvectors of A, A^* are

$$\{\varphi_1, \ldots, \; \varphi_n, \; \varphi_1^*, \ldots, \; \varphi_n^*\}, \; \{\Phi_1, \ldots, \; \Phi_n, \; \Phi_1^*, \ldots, \; \Phi_n^*\}$$
$$A\varphi_j = i\omega_j\varphi_j, \; A\varphi_j^* = -i\omega_j\varphi_j^*, \; A^*\Phi_j = -i\omega_j\Phi_j$$
$$A^*\Phi_j^* = i\omega_j\Phi_j^*, \; (\varphi_j, \; \Phi_j) = (\varphi_j^*, \; \Phi_j^*) = 1, \; j = 1, 2, \ldots, n$$

The projector onto the neutral subspace X_0 is

$$P_0 u = \Sigma \; [(u, \Phi_j)\varphi_j + (u, \Phi_j^*)\, \varphi_j^*]; \; j = 1, 2, \ldots, n$$

The equation in the neutral manifold in the normal form is

$$y_j^{\cdot} = i\omega_j y_j + y_j \Sigma a_{jk} \, | \, y_k \, |^2 + \ldots, \; j, \; k = 1, \; 2, \ldots, \; n;$$
$$y_j = (y', \; \Phi_j)$$
$$a_{jj} = (K_2^0 \, (z_j, \; \varphi_j^*) + K_2^0 \, (\eta_{jj}, \; \varphi_j) + K_3^0 \, (\varphi_j^2, \; \varphi_j^*), \; \Phi_j)$$
$$a_{jk} = (K_2^0 \; (\eta_{kk}, \; \varphi_j) + K_2^0 \, (\eta_{jk}, \; \varphi_k) + K_2^0 \, (h_{jk}, \; \varphi_k^*) +$$
$$K_3^0 \; (\varphi_j, \; \varphi_k, \; \varphi_k^*), \; \Phi_j), \; k \neq j$$
$$z_j = (2i\omega_j I - A)^{-1} K_2\varphi_j^2$$
$$h_{jk} = (i \, (\omega_k + \omega_j) \, I - A)^{-1} \, K_2^0 \, (\varphi_k, \; \varphi_j)$$
$$\eta_{jk} = (i \, (\omega_j - \omega_k) \, I - A)^{-1} \, K_2^0 \, (\varphi_j, \; \varphi_k^*)$$
$$K_3^0 \, (u_1, \; u_2, \; u_3) = \sum_{\pi} K_3 \, (u_{\pi(1)}, \; u_{\pi(2)}, \; u_{\pi(3)})$$

where π is any permutation of the indices 1, 2, 3.

Stability criterion /7/. Co-dimensionality of degeneracy $\mu = n$. The zero equilibrium is stable if the system

$$\rho_j^{\cdot} = \rho_j \Sigma \, (\mathrm{Re} \, a_{jk}) \, \rho_k; \; k, \; j = 1, \; 2, \ldots, \; n$$

has no solutions of the type $\rho_j(t) = c_j r(t)$, $r^{\cdot} = mr^2$, $c_j \geqslant 0$ (not all the $c_j = 0$), neither increasing $(m = 1)$, for neutral $(m = 0)$.

Given any subset $J \subset \{1, 2, \ldots, n\}$, we define the truncation J of the matrix $B = \| \mathrm{Re} \, a_{jk} \|$ $(j, \; k = 1, \; 2, \ldots, \; n)$ as the matrix $B_J = \| \mathrm{Re} a_{jk} \|$, $j, k \in J$. Checking stability amounts to considering all possible systems $B_J \rho_J = e_J$, where all the components of the vector e_J are equal to $+1$. If at least one system has a positive solution $(\rho_j > 0, \; j \in J)$, the equilibrium is unstable. There are altogether $2^n - 1$ such systems, and they must all be considered: suitable examples show that none can be neglected.

If there is no such system, and no system $B_J \rho_J = 0$ has a positive solution, the equilibrium is stable.

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